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# Asymmetric exclusion processes with shuffled dynamics

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**Abstract**

The asymmetric simple exclusion process (ASEP) with periodic boundary conditions is investigated for shuffled dynamics. In this type of update, in each discrete timestep the particles are updated in a random sequence. Such an update is important for several applications, e.g., for certain models of pedestrian flow in two dimensions. For the ASEP with shuffled dynamics and a related truncated process exact results are obtained for deterministic motion ($p = 1$). Since the shuffled dynamics is intrinsically stochastic, also this case is nontrivial. For the case of stochastic motion ($0 < p < 1$) it is shown that, in contrast to all other updates studied previously, the ASEP with shuffled update does not have a product measure steady state. Approximative formulae for the steady-state distribution and fundamental diagram are derived that are in very good agreement with simulation data.

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## 1. Introduction

The asymmetric simple exclusion process (ASEP) is one of the most studied models far from equilibrium [1–3]. It has been used to describe various problems in many fields of research, such as biopolymerization and traffic flow. A lot of analytical results exist, both for open and periodic boundary conditions. The model describes a particle system on a chain with hard core exclusion. Particles are allowed to hop one site to their right, supposing that it is empty. The usual dynamics is in continuous time: the random-sequential update [1]. Moreover, discrete-time update schemes have been studied: backward- and forward-ordered sequential updates, site-oriented [4] as well as particle-oriented [5], sublattice-parallel updates [6, 7] and fully-parallel dynamics [8–10] (for an overview, see [11]). We analyse another update scheme that has originally been introduced in a two-dimensional cellular automaton describing pedestrian dynamics [12, 13].

A typical situation encountered in pedestrian dynamics is the motion along a corridor. This is modelled as a strip of finite width $W$ in $y$-direction of a Cartesian coordinate plane and length $L$ in $x$-direction divided into square cells. Each cell can be in one of two possible states, i.e. either occupied by one of the $N$ pedestrians, or empty (hard core exclusion rule). Assuming periodic boundary conditions in $x$-direction and impenetrable walls in $y$-direction implies that the density of pedestrians $\rho = N/(W \cdot L)$ is constant. A pedestrian can move to neighbour cells with different transition probabilities, depending on the direction and the particular neighbourhood. The preferred direction is along the $x$-axis. A parallel update, in which all particles are updated simultaneously, would lead to conflicts, in which more than one person tries to access the same cell [15]. Hence, another way of updating was chosen, the so-called shuffled dynamics: at each (discrete) timestep, the order in which particles are allowed to move is determined by a random permutation. Since the theoretical implications of this update procedure have not been considered before, we here investigate the one-dimensional limit first. In the language of pedestrian dynamics this corresponds to a very narrow corridor with $W = 1$ such that 'side-by-side' motion or overtaking is not possible. The model then becomes equivalent to the ASEP with shuffled update.

The difference between this update scheme and the random-sequential update mentioned above is analogous to an urn problem with and without replacement: if one imagines an urn with $N$ balls, numbered $1, 2, \ldots, N$, the random-sequential update can be realized by choosing a ball at random, updating the particle with the ball-number and replacing the ball into the urn afterwards. In the shuffled update, one chooses a ball and updates the corresponding particle without replacing the ball into the urn. Then, one chooses the next ball, and so on, until the last particle is updated. Then the urn is refilled and the procedure is repeated in the next timestep.

After giving a precise definition of the model, approximative formulae for the steady-state distribution and fundamental diagram are derived. By mapping onto generalized zero-range processes it is shown that the ASEP with shuffled update does not have a product measure steady state for general parameter choice. Some applications and generalizations are given afterwards and concluding results are discussed and some interesting directions for further research are given.

## 2. ASEP with shuffled dynamics

In the following we give a more formal definition of the ASEP with shuffled dynamics. Consider a one-dimensional lattice with $L$ sites and periodic boundary conditions. Each site may either be occupied by one of the $N$ particles, labelled $i = 1, 2, \ldots, N$, or it may be empty. Therefore the particles are distinguishable. In each discrete timestep a random permutation $\pi\,(1, \ldots, N)$ of the particle labels equals the update sequence. If the right neighbouring cell is empty, the relevant particle moves one site to the right with probability $p$; if it is occupied, the particle stays in its cell.

Figure 1 shows a part of a large system consisting of six cells and four particles (numbered $1, 2, 3, 4$ without loss of generality), at time $t$ (left) and $t + 1$ (right). The drawn update sequence is $\ldots, 3, \ldots, 4, \ldots, 1, \ldots, 2, \ldots,$ where the ellipsis indicates that other particle numbers belonging to different clusters[3] can be chosen in between (for the cluster depicted, only the relative positions of the numbers of its particles in the sequence are of interest). Particle 3, chosen first, cannot move since the cell in front is occupied by 2, and similar for

[3] A cluster is defined as the sequence of occupied cells between two consecutive holes (unoccupied cells).

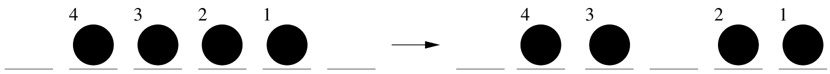

**Figure 1.** Shuffled update of a cluster consisting of four particles numbered from right to left. The drawn sequence is 3, 4, 1, 2 and $p = 1$.

particle 4. Considering the case $p = 1$, particle 1 then moves deterministically to the right. Then 2 also moves, because it was drawn after 1. Although particle 4 is drawn after 3, it cannot move, since both were drawn before 1 and 2.

As pointed out in the introduction, the shuffled update is different from the random-sequential dynamics which is generically used for the ASEP. Whereas the latter describes stochastic processes in continuous time, the shuffled dynamics combines elements of discrete updates and dynamics in continuous time. It is discrete in the sense that there is a well-defined timestep during which each particle is updated exactly once. On the other hand, the order of updating the particles is not fixed. e.g. it may happen that a specific particle is updated last during a timestep and first during the next one! This can actually be a problem for applications because it is difficult to identify the timestep as a kind of reaction time (as it is natural for the case of parallel dynamics [8]).

Despite this important difference, the shuffled and random-sequential updates share certain similarities. In contrast to the ordered updates with fixed order, they do not have a deterministic limit, even for hopping probabilities $p = 1$. However, in the random-sequential case the dynamics depends on $p$ only in a trivial way, since by rescaling time always $p = 1$ can be chosen. This is not possible in the shuffled case. We therefore expect a non-trivial $p$-dependence of the results, as in the other discrete-time updates.

## 3. Steady-state distribution

Using a particle-oriented representation [5, 8], the state of a system with $N$ particles at time $t$ can uniquely be specified by $|n_1(t)^{(\pi_1(t))}, \ldots, n_N(t)^{(\pi_N(t))}\rangle$. Here $n_j$ represents the state of the $j$th particle (i.e. the number of empty sites in front) and $\pi_j$ its update number (the position of the $j$th particle in the ordered update sequence). At each timestep, the permutation operator $\mathcal{P}(t)$ generates the new random sequence as follows: $\mathcal{P}(t)|\ldots, n_i(t)^{(\pi_i(t))}, \ldots, \rangle = |\ldots, n_i(t)^{(\pi_i(t+1))}, \ldots\rangle$, i.e. without changing the variables $n_i(t)$. The complete update of the system $t \to t+1$ can be described by the action of the transfer matrix[4] $h^{(1)}h^{(2)}\ldots h^{(N)}$, being an ordered product of the local operators $h^{(j)}$ acting on the state of the particles $\sigma_j - 1$ and $\sigma_j$: $h^{(j)} = h^{(j)}_{\sigma_j-1,\sigma_j} = \mathbf{1} - p\big(a_{\sigma_j}a^\dagger_{\sigma_j} - a^\dagger_{\sigma_j-1}a_{\sigma_j}\big)$, where $\sigma_j$ is the particle with the update number $j$. This gives the following contribution to the master equation,

$$\langle P|\mathcal{P}(t)\cdot\ldots h^{(i)}\ldots|\ldots, n_i(t)^{(\pi_i(t))}, \ldots\rangle = \langle P|\ldots, n_i(t+1)^{(\pi_i(t+1))}, \ldots\rangle, \tag{1}$$

where $\langle P|\ldots, n_i(t), \ldots\rangle$ denotes the probability of the configuration $|\ldots, n_i(t), \ldots\rangle$. In the steady state, equation (1) simplifies, because all time dependences vanish. Since the permutation operator generates random sequences (each of them with probability $1/N!$) one can calculate the probability with which a particle moves in a certain configuration. The $l$th particle of a cluster has moved at time $t+1$, if and only if at least the first $l$ particles are chosen in the order from the right to the left (with probability $1/l!$) and if they all moved (with probability $p^l$). The probability that *exactly* $l$ particles of a cluster of length $m$ (with $m > l$) move is given as the probability for $l$ particles to move, as calculated above, minus

[4] For the transfer matrices of forward- and backward-sequential update, compare [5].

$p^{k+1}/(k+1)!$, the probability for the $(l+1)$th particle to move. We arrive at $u_l(m)$, the probability for $l$ particles leaving a cluster of length $m$ (where $0 \leqslant l \leqslant m$):

$$u_l(m) = \frac{p^l}{l!} - \frac{p^{l+1}}{(l+1)!}\theta(m-l) \tag{2}$$

(and 0 otherwise). The Heaviside step function ensures that in the case $l = m$ the right term vanishes. Note that although the particles in the ASEP are updated in $N$ steps, it can be considered as if all clusters were updated in parallel.

We now leave the operator notation and write the steady-state probability (calculated at the end of the timestep when all particles have been updated) simply as $P(n_1, n_2, \ldots, n_N)$. We assume that this probability can be written as a product of the probabilities $P_{n_i}$ to find particle $i$ with $n_i$ holes in front, i.e.

$$P(n_1, n_2, \ldots, n_N) = P_{n_1} P_{n_2} \ldots P_{n_N}. \tag{3}$$

This constitutes the so-called car-oriented mean-field (COMF) theory, successfully applied previously to traffic flow models [22, 23]. Note that equation (3) implies that only correlations between neighbouring particles are taken into account. In the thermodynamic limit, the master equation for the steady state reads:

$$P_0 = P_0 - (p-g)(1-P_0) + p\bar{g}P_1, \tag{4}$$

$$P_1 = (p-g)(1-P_0) + (pg + \bar{p}\bar{g})P_1 + p\bar{g}P_2, \tag{5}$$

$$P_n = \bar{p}gP_{n-1} + (pg + \bar{p}\bar{g})P_n + p\bar{g}P_{n+1}, \qquad \forall n > 1. \tag{6}$$

Here $P_k$ $(k = 0, 1, \ldots)$ is the probability for an arbitrary particle to have $k$ holes (a hole-cluster of length $k$) in front and $g = 1 - \bar{g}$ is the probability that the particle in front moves. Note that the sum of the rhs of equations (4)–(6) indeed gives $\sum_n P_n$. As an example we explain the equation for $P_1$ in the following: a gap of 1 hole can arise from three different processes corresponding to the three terms on the rhs of (5): if the particle has 2 holes in front (probability $P_2$) the gap decreases by 1 if the particle itself moves (probability $p$), but the preceding one not (probability $\bar{g}$). The gap remains unchanged (probability $P_1$) if either both particles move (probability $pg$) or if both particles do not move (probability $\bar{p}\bar{g}$). If the particle has its preceding particle directly in front (probability $P_0$) the situation is more sophisticated: the particle has one hole in front afterwards if and only if the particle in front has moved, but the particle itself has not. This probability depends on the position of the particle in the cluster to which itself and the preceding particle belong. The probability $P(k)$ for the particle to have exactly $k-1$ particles directly in front is approximated by

$$P(k) = P_0^{k-1}(1-P_0). \tag{7}$$

This probability (7) has to be multiplied with $u_{k-1}(n) = \frac{p^{k-1}}{(k-1)!} - \frac{p^k}{k!}$, for $k \leqslant n$, obtained from equation (2). Finally we have to sum over all possible $k$. This yields $(1-P_0)\sum_{k=2}^{\infty}\left(\frac{p^{k-1}}{(k-1)!} - \frac{p^k}{k!}\right)P_0^{k-1}$ which can be rewritten as the first term on the rhs of (5) as one can check easily if one interprets $g$ as the hopping probability for a particle, averaged over all possible numbers of particles in front, i.e.

$$g = \sum_{k=1}^{\infty} \frac{p^k}{k!} P(k) = \begin{cases} p, & \text{for} \quad P_0 = 0, \\ \dfrac{1-P_0}{P_0}(\exp(pP_0) - 1), & \text{for} \quad P_0 > 0. \end{cases} \tag{8}$$

Using this, the system of equations (4)–(6) can be solved by generating functions [22, 23]. We obtain an implicit expression for $P_0$ that has to be solved numerically in general, since $g$ and $\bar{g}$ depend on $P_0$ via (8):

$$P_0 = \frac{p(\rho - \bar{\rho}) - (p\rho - \bar{\rho})g}{p\rho\bar{g}}, \tag{9}$$

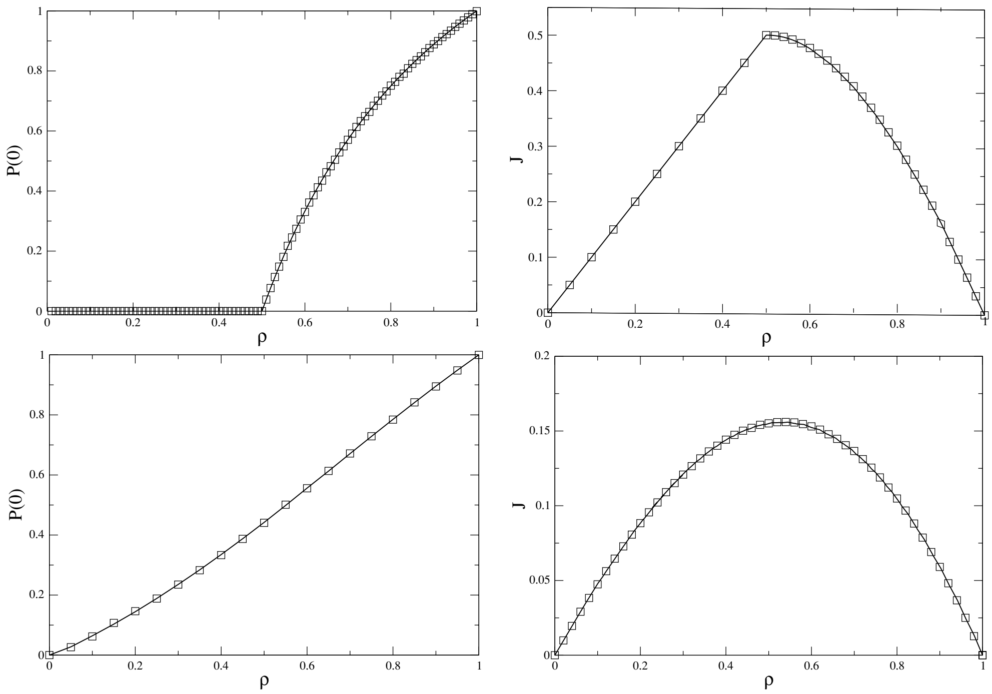


**Figure 2.** The probability $P_0(\rho, p)$ (left) and the fundamental diagram $J(\rho, p)$ (right) for $p = 1$ (top) and $p = 0.5$ (bottom). Depicted are the results from COMF (lines) and from computer simulations (squares) for a system consisting of $L = 500$ cells. Note that $P_0(\rho, p = 1)$ is exact.

$$P_n = \frac{(p-g)(1-P_0)}{\bar{p}g}\left(\frac{\bar{p}g}{p\bar{g}}\right)^n, \quad \forall n > 0. \tag{10}$$

Again, the abbreviations $\bar{\rho} = 1 - \rho$, etc were used. Since $g$ is the probability that an arbitrary particle moves, it equals also the average velocity $\bar{v}$ of the particles. This is related to the flow $J$ and yields the so-called fundamental diagram

$$J(\rho) = \bar{v}\rho = g\rho. \tag{11}$$

In the (partially deterministic) case $p = 1$, $P_0$ becomes

$$P_0 = \frac{2\rho - 1}{\rho}\theta\left(\rho - \frac{1}{2}\right), \tag{12}$$

as one can check easily. Note that $p = 1$ and $\rho \leqslant 1/2$ is the only nontrivial case in which $P_0$ can vanish. For $p = 1$, the probability $P_0$ is completely determined by the fact that the first particle of a cluster moves deterministically and all the other particles have smaller hopping probabilities due to the shuffling. For densities $\rho \leqslant 1/2$ this implies that any state which consists only of clusters of size 1, i.e. separated particles, is stationary. Hence the probability to find a particle directly in front vanishes and we have $P_0 = 0$ (figure 2). For densities $\rho > 1/2$ clusters are formed that are separated by exactly one hole, i.e. there are only isolated empty cells in the steady state. It is easy to verify that no pairing of holes can happen, since from the point of view of the holes they jump at least one site backwards and at most to the end of the cluster. As a consequence, $P_n = 0$ for $n \geqslant 2$. Therefore $L - N$ clusters exist, which is then also the number of particles having exactly one hole in front. Thus we obtain

$P_1 = (1-\rho)/\rho$ and $P_0 = 1 - P_1 = (2\rho-1)/\rho$ in this density regime. These results are exact for $p = 1$ and are reproduced by COMF (see (9)–(12)).

The calculation of the fundamental diagram using (11) requires the knowledge of $g$. This in turn depends on the cluster length distribution which is not known exactly. Using the approximation (11), the flow–density relation is explicitly given by

$$J(\rho, p=1) = \begin{cases} \rho, & \text{for} \quad \rho \leqslant 1/2, \\ \dfrac{\rho(1-\rho)}{2\rho-1}\left[\exp\left(\dfrac{2\rho-1}{\rho}\right)-1\right], & \text{for} \quad \rho > 1/2. \end{cases} \tag{13}$$

The fundamental diagram (13) shows a strong asymmetry with respect to $\rho = 1/2$. For densities $\rho \leqslant 1/2$ each particle can move independently and deterministically for $\rho \leqslant 1/2$, since every particle has at least one hole in front, exactly as in parallel updating [23]. If the density is increased to values greater than $1/2$, $L - N$ clusters of nonvanishing length are formed from which the rightmost particle can move deterministically and since the probability to find such a particle in the system is given by $1-\rho$, they add exactly this value to the flow, as in the usual parallel update. Consequently, this result can be obtained from (13) by a first-order Taylor expansion. The contribution of the other particles to the flow depends exponentially on the ratio of these particles $(\rho - (1-\rho))/\rho = (2\rho-1)/\rho$. This yields the curvature in the fundamental diagram in the high density regime. In contrast to the parallel and random-sequential dynamics, the shuffled update is not particle–hole symmetric (figure 2).

In this partially deterministic case $p = 1$ clearly a free-flow phase ($\rho < 1/2$) and a jammed phase ($\rho > 1/2$) can be distinguished. They are separated by a phase transition at $\rho_c = 1/2$. The probability $P_0(\rho)$ constitutes a kind of order parameter. It vanishes exactly for $\rho \leqslant 1/2$ and increases continuously for $\rho > 1/2$ (figure 2) indicating a second-order transition. This is similar to the deterministic limit of parallel dynamics [26]. With decreasing hopping probability the asymmetry of the fundamental diagram becomes smaller. In the limit $p \to 0$ we obtain the result for random-sequential update, $J = p\rho(1-\rho)$. For $p < 1$ the free-flow and jammed regimes are no longer separated by a phase transition as can be seen from the smooth behaviour of $P_0(\rho)$ (figure 2).

We just mention [24] that the assumption of a factorized steady state in a finite system yields less agreement, but the analytical results converge fast to the result of the thermodynamic limit, derived here. In the case $p = 1$, for example, the largest possible cluster has length $2N - L + 1$ and the term $\frac{\rho}{2\rho-1}\big[\exp\big(\frac{2\rho-1}{\rho}\big)-1\big]$ in (13) has to be replaced by the generalized hypergeometric function ${}_2F_2(1, L-2N; 2, 1-N; 1)$ [25], which overestimates the results for small $L$, indicating that the mean-field result is not exact, at least for finite systems.

## 4. Mapping onto a generalized zero-range process with parallel dynamics

The ASEP can be mapped onto a model with multi-occupation of sites, the generalized zero-range process (GZRP). Unoccupied cells, i.e. holes, of the ASEP become the sites of the GZRP and the particles between the $(i-1)$th and the $i$th hole of the ASEP (the particles of the $(i-1)$th cluster) now all occupy the $i$th site of the GZRP and are referred to as mass $m_i$ (see figures 3(*a*) and (*b*)). We have $L - N$ sites and masses and still $N$ particles. At each timestep, $l_i$ particles are chipped off the mass $m_i$ located at site $i$ $(i = 1, \ldots, L-N)$ with transition probabilities $u_{l_i}(m_i)$ and are moved to site $i+1$. In the ASEP with shuffled dynamics the probabilities $u_l(m)$ are given by (2). Note that although the particles in the ASEP are updated in $N$ steps, it can be considered as if all *clusters* were updated in parallel. Thus indeed the ASEP with shuffled update is mapped onto a GZRP with parallel dynamics. In the case $p = 1$

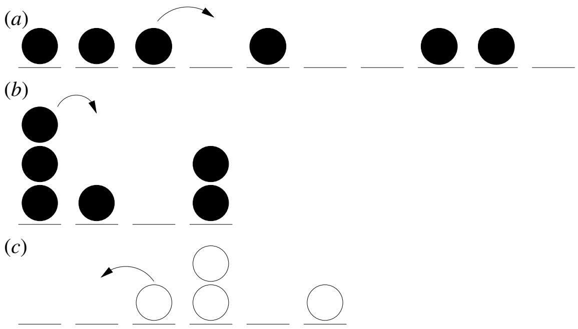


**Figure 3.** Mapping of the ($a$) ASEP onto ($b$) GZRP and ($c$) ZRP for $L = 10$ and $N = 6$. The arrows indicate a possible local transition with probability $u_1(3)$.

and $N < L - N$, we have seen that in the ASEP the probability to find two neighbouring particles vanishes. For the GZRP this implies that each cell is either occupied by exactly one particle, or it is empty. Thus the GZRP is, in fact, a usual ZRP in this case and its steady-state distribution can be written as a product measure.

The authors of [19] have derived a necessary and sufficient condition for the existence of a factorized steady-state distribution in a broad class of one-dimensional mass transport models, including the GZRP. This is possible if the transition probabilities $u_l(n)$ can be written as a product $v_l w_{n-l} / \sum_{l=0}^{n} v_l w_{n-l}$, where $v_l$ and $w_{n-l}$ are functions that depend only on $l$ and $n-l$, respectively. They also derived a more direct test in [21]. For continuous-time dynamics a related condition has been found recently for processes in arbitrary dimensions [20]. The transition probabilities (2) of the corresponding GZRP do not satisfy this condition for general $p$ and $\rho$ and therefore the steady state does not factorize, i.e.

$$\tilde{P}(m_1, m_2, \ldots, m_{L-N}) \neq \tilde{P}_{m_1} \tilde{P}_{m_2} \cdots \tilde{P}_{m_{L-N}}. \qquad (14)$$

In the ASEP, this implies that the particle-cluster probabilities do not factorize. However, a factorization into hole-cluster probabilities as assumed in COMF is not excluded.

## 5. Mapping onto a zero-range process with shuffled dynamics

While in the GZRP an arbitrary fraction of particles is allowed to chip off a mass, in the usual zero-range process (ZRP) [16] at most one particle (say the topmost) can leave a mass during one timestep. The ASEP can be mapped onto a ZRP [17, 18] by identifying the particles of the exclusion model with the sites of the zero-range process (see figures 3($a$) and ($c$)). The holes between the $i$th and the $(i + 1)$th particle of the ASEP form the $i$th mass of the ZRP. Thus the ZRP has $N$ sites and $L - N$ particles. Note that the particles in the ZRP hop to the left, as the holes in the ASEP. While in the ASEP the particles were shuffled, in its equivalent ZRP the sites are shuffled. Zero-range processes are known to have a factorized steady-state distribution [17, 18]. This result holds for time-continuous dynamics and for parallel and ordered-sequential updating [5] as well. For the shuffled update no general results are available yet. The COMF approach in section 3 (equation (3)) describes a factorization in the ZRP picture. For $p = 1$, the result (12) is exact. However, it is not clear whether the fundamental diagram is exact or not, since it depends also on $g$ which we do not know exactly.

It can easily be seen that this factorization does not become exact for general $p$, by considering the master equation for the probability $P(0^{N-1}, M)$, i.e. one cell of the ZRP having occupation number $M := L - N$, followed by $N - 1$ empty sites (a cluster of $N$ particles followed by $L - N$ holes in the ASEP):

$$P(0^{N-1}, M) = \left(\bar{p} + \frac{p^N}{N!}\right) P(0^{N-1}, M) + \bar{p} \sum_{k=1}^{N-1} \frac{p^k}{k!} P(0^{N-k-1}, 1, 0^{k-1}, M-1), \tag{15}$$

with $\bar{p} = 1 - p$. Inserting the product measure ansatz (3) and performing the limit $N \to \infty$ gives $pP_0P_M = \bar{p}P_1P_{M-1}(e^p - 1)$. Now, inserting the expression for $P_n, n \geqslant 1$ from equation (10) and using (8) leads to the constraint

$$0 = (e^p - 1)\left(\frac{p}{e^{pP_0} - 1} - \frac{1 - P_0}{P_0}\right) - p. \tag{16}$$

This condition is only fulfilled in the limit $P_0 = 1$ (i.e. for $\rho = 1$) and also for $p = 0$. Thus one can conclude that the COMF theory gives a good approximation but not the exact result, for general $p$. In other words, the ZRP with shuffled update does not have a product measure steady state.

## 6. Truncated processes

In section 3 we have seen through mapping onto a GZRP that the shuffled update can also be interpreted as a cluster dynamics. The probabilities that $l$ particles leave a cluster of length $m$ are given by (2). These probabilities decrease rapidly with increasing $l$. Therefore it is natural to consider as an approximation to the original dynamics models in which, irrespective of the cluster length, at most $l_{\max}$ particles are allowed to leave a cluster at any given timestep. This truncation leads to a GZRP defined by probabilities

$$u_l^{(l_{\max})}(m) = \frac{p^l}{l!} - \frac{p^{l+1}}{(l+1)!}\theta(l_{\max} - l), \tag{17}$$

with $0 \leqslant l \leqslant \min(l_{\max}, m)$ (and 0 otherwise). The case $l_{\max} = 1$ corresponds to the standard ZRP with hopping probability $u(m) = p$, i.e. the ASEP with parallel dynamics. Thus, the fundamental diagram is given by $J_1(\rho, p) = \frac{1}{2}(1 - \sqrt{1 - 4p\rho(1-\rho)})$ [8].

In the case $l_{\max} = 2$ the factorization condition of [19, 21] is satisfied for $p = 1$ and we have $\tilde{P}(m_1, m_2, \ldots, m_{L-N}) = \tilde{P}_{m_1}\tilde{P}_{m_2} \cdots \tilde{P}_{m_{L-N}}$. For the fundamental diagram we obtain

$$J_2(\rho, p = 1) = \begin{cases} \rho, & \text{for} \quad \rho \leqslant 1/2, \\ 1 - \dfrac{\rho + \sqrt{\rho^2 - 2(1-\rho)(2\rho - 1)}}{2}, & \text{for} \quad \rho > 1/2. \end{cases} \tag{18}$$

Figure 4 shows the results for $l_{\max} = 1, 2$ in comparison with the shuffled update $l_{\max} = \infty$. The steady-state probabilities for $l_{\max} = 2$ depend on the number of particle-clusters with length 1. This is not correctly reproduced by COMF theory which assumes a factorization into hole-cluster probabilities (3) and yields that all stationary configurations for fixed particle number $N$ are equally probable for $p = 1$ and $\rho > 1/2$: $P(n_1, n_2, \ldots, n_N) = P_0^{2N-L}P_1^{L-N}$. Therefore, one can conclude that the steady-state distribution for the truncated model with $l_{\max} = 2$ and $p = 1$ factorizes into particle-cluster probabilities but not into hole-cluster probabilities.

For $l_{\max} = 3, 4, \ldots$, the curves converge quickly to the shuffled curve, but they no longer fulfil the factorization criterion.

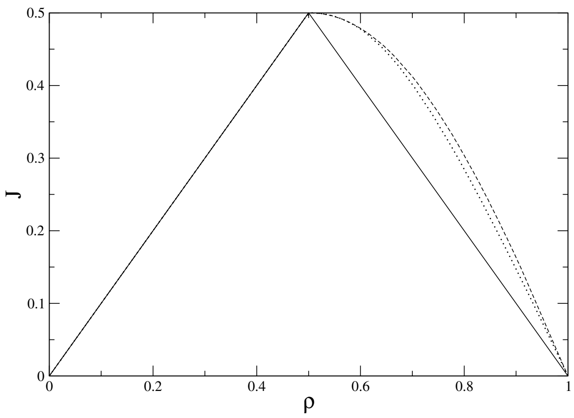


**Figure 4.** Comparison of truncated models for $p = 1$. The continuous curve shows the case $l_{\max} = 1$, i.e. the case of usual parallel update. The dotted curve shows $l_{\max} = 2$, given by (18) and the dashed curve shows the fundamental diagram for the shuffled update, obtained for $l_{\max} = \infty$.

## 7. Application to pedestrian dynamics

As mentioned at the beginning, the shuffled update has an interpretation [12, 13] as an update procedure in two-dimensional cellular automata to avoid conflicts (situations in which more than one particle tries to access the same cell) occurring in parallel dynamics [15]. An important example is the movement of pedestrians in a long corridor. To describe this movement more realistically we consider two straightforward generalizations of the model in the following. The simplest extension of the ASEP, which corresponds to a single-lane model is a model of several decoupled lanes. Here pedestrians are not allowed to change lane, i.e. their $y$-coordinates are fixed. If in the initial state the pedestrians are distributed stochastically on sites (and thus lanes), their density in each lane is fixed and constant over time. The probability of finding $i$ pedestrians in a particular lane is given by the hypergeometric distribution

$$\mathrm{Hyp}_{N,L,L(W-1)}(i) = \binom{L}{i}\binom{L(W-1)}{N-i}\Big/\binom{LW}{N} \tag{19}$$

with mean $\rho = N/(LW)$. The fundamental diagram of the model with decoupled lanes is [27]

$$\tilde{J}(\rho) = \sum_{i=0}^{L} \mathrm{Hyp}_{\rho LW,L,L(W-1)}(i) \cdot J(i/L), \tag{20}$$

where $J(\rho)$ is the single-lane fundamental diagram determined in section 3. Using the approximation (11), the result for a width of $W = 10$ cells and deterministic hopping ($p = 1$) in the thermodynamic limit is depicted in figure 5($a$). One can see that the maximum is smoother and a little bit lowered in comparison to the corresponding figure 2($b$). Note that lane-changing, an effect that is especially relevant for high densities, would lead to a reduction of the flow. To reproduce the shift of the maximum flow to lower densities, observed in pedestrian dynamics, we introduce a larger maximum velocity $v_{\max}$, i.e. a larger number of cells that can be passed during one timestep [28]. The simplest way is to allow a pedestrian to move the minimum of $v_{\max}$ cells and the number of empty cells in front with probability $p$ at each timestep. In the following we again consider the case $W = 1$ and $p = 1$ only. For densities less than or equal to $1/(v_{\max} + 1)$ each of the pedestrians has at least $v_{\max}$ empty sites

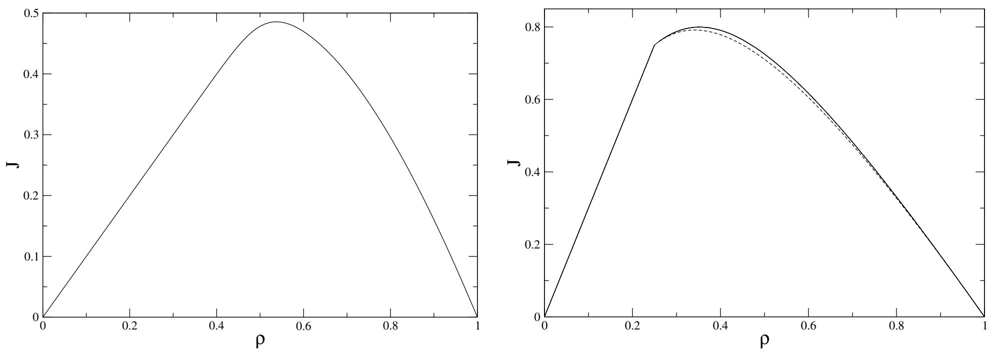


**Figure 5.** Fundamental diagrams for two generalized processes with $p = 1$. The left diagram shows the result for the model with decoupled lanes for $W = 10$. The right one shows the case of increased velocity $v_{\max} = 3$ for $W = 1$. The dashed curve represents the analytical result, the continuous curve the result from computer simulations.

in front, i.e. the probability to find a particle in front is $P_0 = 0$ and the flow is deterministically given as $v_{\max}\rho$. For higher densities it may happen that, e.g. a pedestrian occupies a cell at time $t + 1$ that has been occupied by a different pedestrian belonging to the cluster in front at time $t$. However we found in computer simulations that this effect can be neglected (for relatively small $v_{\max}$). Thus one can again describe the dynamics in the picture in which all clusters are updated in parallel. The system tries to separate the clusters by exactly $v_{\max}$ holes. The probability to have at least one hole in front is in this case $(1 - \rho)/v_{\max}$ and the ratio of particles which do not have a particle in front is $(\rho - (1 - \rho)/v_{\max})/\rho$. This defines the probability $P_0$, which can finally be written as $[(v_{\max} + 1)\rho - 1]\theta(\rho - 1/(v_{\max} + 1))/(v_{\max}\rho)$. With the use of (13) we obtain

$$J(\rho, v_{\max}, p=1) = \begin{cases} v_{\max}\rho, & \text{for } \rho \leqslant 1/(v_{\max}+1), \\ \dfrac{v_{\max}\rho(1-\rho)}{(1+v_{\max})\rho - 1}\left[\exp\left(\dfrac{(v_{\max}+1)\rho - 1}{v_{\max}\rho}\right) - 1\right], & \text{else.} \end{cases} \tag{21}$$

In figure 5(*b*) the resulting fundamental diagram in the case of $v_{\max} = 3$ and $p = 1$ is depicted. The dashed line shows the analytical in comparison to the numerical result. In opposite to the case of maximum velocity 1, depicted in figure 2(*b*), the maximum of the fundamental diagram and the critical point $(1/(v_{\max} + 1), \rho v_{\max})$ do not coincide; this is qualitatively reproduced by the analytical result. However, to recover the exact result one would have to take longer ranged correlations into account.

## 8. Discussion

In this paper, the asymmetric exclusion process (ASEP) with periodic boundary conditions and shuffled dynamics was studied. Using a mean-field approach, steady-state properties such as the fundamental diagram and distribution functions are derived that are in very good agreement with data from Monte Carlo simulations of large systems.

In the case $p = 1$ the exact expression for the probability $P_0$ that a particle has no hole in front is found, being the same as in usual parallel updating. Since the shuffled dynamics is intrinsically stochastic, we found already for $p = 1$ a nontrivial fundamental diagram (depicted in figure 2(*b*)). Despite the stochasticity of the update surprisingly in the regime

$\rho \leqslant 1/2$ all particles move deterministically in the steady state. The stationary flow is then identical to the deterministic limit of the free flow state for a deterministic parallel update and could be calculated exactly. For higher densities $\rho > 1/2$ the probability for a particle to move depends on the number of particles in front and therefore the flow depends on the cluster-size distribution which is not known exactly. However, mean-field theory yields a good approximation for the flow. The two regimes are separated by a second-order phase transition. In the case $p < 1$ the two regimes can no longer be distinguished and the flow is determined by the stochasticity for all densities. The fundamental diagrams become smooth, but still do not exhibit a particle–hole symmetry. The free flow and jammed regimes are no longer separated by a phase transition. In the limit $p \to 0$ we obtained the same result as for random-sequential updating.

By mapping onto a generalized zero-range process with parallel update and using a test for a factorized steady state derived recently [19], it could be shown that the ASEP with shuffled update surprisingly does not factorize into particle-cluster probabilities. Furthermore, we found that it also does not factorize into hole-cluster probabilities. This implies that the ZRP with shuffled update is not solved by a product measure. This is the first example of a non-factorizing ZRP, since all other updates have led to a product measure steady state in the ZRP formulation so far. In contrast to the other updates investigated so far it is essential that we are dealing with distinguishable particles. The numbering is needed in order to define the update. This is not the case for random-sequential, parallel or ordered-sequential updates. Note that in the random-sequential case sites could be updated instead of particles without changing the stationary state. For the ordered-sequential dynamics only one particle has to be marked, namely the first one of the sequence.

Further, a truncated process was presented in which at most two particles of a cluster can move and presented its exact solution for $p = 1$. We could show that this model factorizes into particle-cluster probabilities but not into hole-cluster probabilities. Since in shuffled dynamics the probability that the third particle in a cluster moves is relatively small, the truncated dynamics gives also a good approximation for the model with shuffled update.

It has been shown [12, 13] that cellular automata with shuffled update have an interpretation as pedestrian flow models. In this sense the ASEP with shuffled update models the directed motion of pedestrians in a long corridor of small width. To allow pedestrians to walk side by side, we generalized the model to a two-dimensional scenario with decoupled lanes, in which pedestrians cannot change lanes. Further we gave a generalization to higher maximum velocities. For both generalizations we presented approximations that are in good agreement with Monte Carlo data.

We want to mention that the idea of ordered sequential updates (forward- or backward-ordered sequences) can be generalized [29]: consider a general ordered sequential update with sequence $\pi = (\pi(1), \pi(2), \ldots, \pi(N))$, which now is quenched for all timesteps. $\pi(1)$ denotes the update number of the particle with number 1 and so on, where the particles are numbered from left to right. The forward sequential dynamics is the special case of $\pi_F = (1, 2, \ldots, N)$ and the backward sequential dynamics is the special case of $\pi_B = (N, N-1, \ldots, 1)$. As far as we know, other update sequences have not been studied yet. However, it is easy to show that for $N$ even all $((N/2)!)^2$ sequences, in which the even (odd) particles are updated first, and then the odd (even) particles, have a factorized steady-state distribution [24]. Note that these updates are equivalent to a sublattice-parallel-like update, in which in the first half of a timestep all even (odd), and in the second half, all odd (even) particles are updated in parallel. However, the usual ordered sequential update does not factorize with a general update sequence. Hence, it would be interesting to find a condition for a model with sequence $\pi$ (and transfer matrix $h^{(1)} \cdots h^{(N)}$) to have a factorized steady state. Among that, if an update with sequence $\pi$

generates the steady state $F(\pi)$, it is interesting to consider the average of the steady states generated by all permutations of the permutation group $S$: $\langle F\rangle = \sum_{\pi\in S} F(\pi)/N!$. We found in computer simulations that this steady state differs from the steady state of the shuffled update distinctly, what cannot be trivially anticipated.

Summarizing, maybe the most surprising result is the fact that the ASEP with shuffled update does not factorize, in contrast to the updates investigated before. Here the distinguishability of the particles seems to be important whereas in the updates considered previously the particles are basically indistinguishable. This point warrants further investigations.

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